Lecture 16

Infinite Sets & Countability

Some Theorems On Finite Sets

Recall:

Definition: Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say that S is a finite set and that n is the **cardinality** of S. The cardinality of S is denoted by $|S|$. A set is said to be **infinite** if it is not finite.

Let A and B be two finite sets.

Theorem: There is an injunction from A to B if and only if $|A| \leq |B|$.

Theorem: There is a surjection from A to B if and only if $|A| \geq |B|$.

Theorem: There is a bijection from A to B if and only if $|A| = |B|$.

Will use them to extend the comparison of cardinality of all sets (finite + infinite).

Comparing Cardinalities

Can be finite or infinite

Definition: Two sets A and B have the **same cardinality** if and only if there is a bijection from A to B. We write $|A| = |B|$, when A and B have the same cardinality.

Definition: There is an injunction from A to B if and only if the cardinality of A is **less than or the same as the cardinality** of B. We write $|A| \leq |B|$, in such a case.

Definition: When there is an injunction from A to B but no bijection from A to B, we say the cardinality of A is less than the cardinality of B and we write $|A| < |B|$.

Two Kinds of Infinite Sets

Infinite Sets

Sets with the same cardinality as $\mathbb{Z}^+$

Examples:

- $\mathbb{Z}$, set of integers.
- $\mathbb{Q}$, set of rational numbers.
- Set of even numbers.

Sets with cardinality different from $\mathbb{Z}^+$

Examples:

- $\mathbb{R}$, set of real numbers.
- Set of real numbers between 0 and 1.
- Power set of $\mathbb{Z}^+$.

Countable Sets

Elements of some sets can be listed out as 1st element, 2nd element, 3rd element, etc.

- In $\{a, e, i, o, u\}$, we can call a as 1st element, e as 2nd element, i as 3rd element, etc.
- In $\mathbb{O}^+$, we can call 1 as 1st element, 3 as 2nd element, 5 as 3rd element, etc.

Definition: A set that is either finite or has the same cardinality as $\mathbb{Z}^+$ is called **countable**.
A set that is not countable is called **uncountable**.

Examples: Countable Sets

Example: Show that $\mathbb{O}^+$, i.e., set of odd positive integers, is a countable set.

Solution: We need to give a bijection from $\mathbb{Z}^+$ to $\mathbb{O}^+$.

Let $f(n) = 2n - 1$ be a function from $\mathbb{Z}^+$ to $\mathbb{O}^+$. We prove now that f is a bijection.

f **is onto:**

Let k be an odd positive integer. Then, for $n = \dfrac{k+1}{2}$, $f(n) = k$.

f **is one-to-one:**

Suppose f is not one-to-one and $\exists\, n_1, n_2$ such that $n_1 \neq n_2$ and $f(n_1) = f(n_2)$.

$f(n_1) = f(n_2) \implies 2n_1 - 1 = 2n_2 - 1 \implies n_1 = n_2$, which is a contradiction.

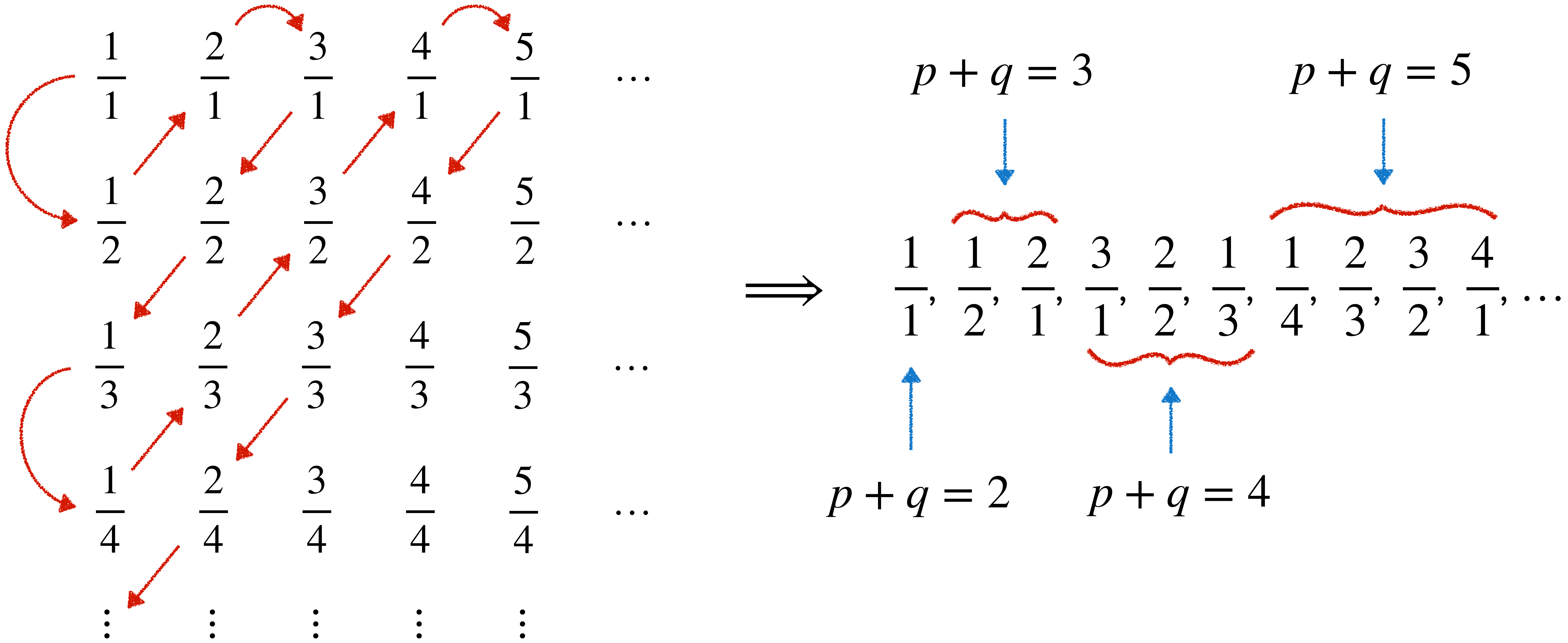
Hence, f is one-to-one. ■

Examples: Countable Sets

Example: Show that $\mathbb{Q}^+$, i.e., set of positive rational numbers, is a countable set.

Solution: It is enough to list elements of $\mathbb{Q}^+$ s.t. every $a \in \mathbb{Q}^+$ appears exactly once. (Why?)

Let's arrange all the positive rational numbers in an infinite 2-dimensional matrix.



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